1

I/ $I(b,a)=\int_0^1 x^{b-1}(1-x)^{a-b}dx$ où $1\leq b\leq a$.

Q.1a : $I(1,a)=\int_0^1 (1-x)^{a-1}dx=\frac{1}{a}$.

Q.1b : On effectue une IPP

$$I(b+1,a)=\int_0^1 x^b(1-x)^{a-b-1}dx=\left[\frac{(1-x)^{a-b}}{b-a}x^b\right]_0^1+\frac{b}{a-b}I(b,a)$$

Finalt : $\boxed{I(b+1,a)=\frac{b}{a-b}I(b,a)}$

Q1c/ $I(b,a)=\frac{b-1}{a-b+1}I(b-1,a)=\frac{(b-1)!}{(a-b+1)\cdots(a-1)}I(1,a)$

D'où $I(b,a)=\frac{(b-1)!(a-b)!}{a!}=\frac{1}{b}\times\frac{1}{\binom{a}{b}}$

Q2a/ $\int_0^1(1-x+xy)^{a-1}dx=\int_0^1\sum_{i=0}^{a-1}\binom{a-1}{i}x^iy^i(1-x)^{a-1-i}dx$.

$$=\sum_{i=0}^{a-1}\binom{a-1}{i}y^i\int_0^1 x^i(1-x)^{a-1-i}dx \qquad (k=i+1)$$

$$=\sum_{k=1}^{a}\binom{a-1}{k-1}y^{k-1}I(k,a)$$

Q2b/ $\int_0^1(1+(y-1)x)^{a-1}dx=\frac{1}{a(y-1)}\left[(1+(y-1)x)^a\right]_0^1$

$$=\frac{y^a-1}{a(y-1)}=\frac{1}{a}\sum_{k=1}^{a}y^{k-1}$$

Q2c : en identifiant les coef. des polynômes en y

on obtient $\frac{1}{a}=\binom{a-1}{k-1}I(k,a)$

ie $\boxed{I(k,a)=\frac{1}{a\binom{a-1}{k-1}}=\frac{1}{k\binom{a}{k}}}$ //

Q3a/

$$I(b,a)=\int_0^1 x^{b-1}\sum_{k=0}^{a-b}\binom{a-b}{k}(-1)^k x^k\,dx$$

$$=\sum_{k=0}^{a-b}\binom{a-b}{k}(-1)^k\int_0^1 x^{k+b-1}\,dx$$

Finalt $\boxed{I(b,a)=\sum_{k=0}^{a-b}\binom{a-b}{k}(-1)^k\frac{1}{k+b}}$

Q3b/ $\Delta_a=\text{ppcm}(1,2,\dots,a)$

$$\Delta_a I(b,a)=\sum_{k=0}^{a-b}\binom{a-b}{k}(-1)^k\times\frac{\Delta_a}{k+b}$$

or $k\in[\![0,a-b]\!]$ d'où $k+b\in[\![b,a]\!]$ et $\frac{\Delta_a}{k+b}\in\mathbb{N}$

Finalt $\Delta_a I(b,a)\in\mathbb{Z}$ or $\Delta_a\geq 0$ et $I(b,a)\geq 0$

Donc $\boxed{\Delta_a I(b,a)\in\mathbb{N}}$

Q3c/ $I(b,a)\,\Delta_a=\frac{\Delta_a}{b\binom{a}{b}}\in\mathbb{N}$

Donc $\boxed{b\binom{a}{b}\text{ divise }\Delta_a}$

Q4a/ en prenant $a=2n$ et $b=n$

$n\binom{2n}{n}$ divise Δ_{2n} or $\Delta_{2n}\,/\,\Delta_{2n+1}$

Finalt $\boxed{n\binom{2n}{n}\,/\,\Delta_{2n+1}}$

$(2n+1)\binom{2n}{n}=\frac{(2n+1)!}{n!\,n!}=(n+1)\binom{2n+1}{n+1}$ donc divise Δ_{2n+1}

ie $\boxed{(2n+1)\binom{2n}{n}\,/\,\Delta_{2n+1}}$

Q4b/ $\Delta_{2n+1} = n\binom{2n}{n} k = (2n+1)\binom{2n}{n} k'$

donc $nk = (2n+1)k'$ or $n \wedge (2n+1) = 1$

Par le th de Gauss : n divise k'

cà $\Delta_{2n+1} = n(2n+1)\binom{2n}{n} k''$

cà $\boxed{n(2n+1)\binom{2n}{n} \mid \Delta_{2n+1}}$

Q5a/ Posons $u_{n,k} = \binom{2n}{k} = \frac{2n!}{k!(2n-k)!} > 0$

$$\frac{u_{n,k+1}}{u_{n,k}} = \frac{2n-k}{k+1}$$

donc $(u_{n,k})_k \nearrow$ ssi $2n-k \geq k+1$ ssi $k \leq n - \frac{1}{2}$

cà $(u_{n,k})_k \nearrow$ pour $k \in [0,n]$ et $\searrow$ pour $k \in [\![n, 2n]\!]$

cà $u_{n,k} \leq u_{n,n}$ ou encore $\boxed{\binom{2n}{k} \leq \binom{2n}{n}}$

Q5b/ $4^n = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \leq (2n+1)\binom{2n}{n}$

$n(2n+1)\binom{2n}{n}$ divise Δ_{2n+1} or $(2n+1)\binom{2n}{n} \geq 4^n$

Donc $\boxed{\Delta_{2n+1} \geq n4^n}$

Q6 $m \geq 9$

1er cas $m = 2n+1$ $(n \geq 4)$

$$\Delta_m \geq n4^n = \frac{m-1}{2} \cdot 2^{m-1} \geq 4 \cdot 2^{m-1} > 2^m$$

2e cas $m = 2n$ $(n \geq 5)$

$$\Delta_m \geq \Delta_{m-1} = \Delta_{2n-1} = \Delta_{2(n-1)+1} \geq (n-1)4^{n-1} > 2^m$$

Rem/ si $m = 7$ $\Delta_7 = 420 > 2^7 = 128$ or $\Delta_6 = 60 < 2^6 = 64$

si $m = 8$ $\Delta_8 = 840 > 2^8 = 256$

Q7/ $n \geq 1$ et $p \in \mathcal{P}$

$v_p(\Delta_n) = \max(v_p(1), v_p(2), \dots, v_p(n))$

donc $\exists k \in [\![1, n]\!]$ tq $v_p(\Delta_n) = v_p(k)$

si $\exists p \in \mathcal{P}$ tq $p^{v_p(\Delta_n)} > n$ on aura $p^{v_p(k)} > n$

or $k = \prod q^{v_q(k)} \geq p^{v_p(k)} > n$ impossible

Finalt : $\forall p \in \mathcal{P}$, $p^{v_p(\Delta_n)} \leq n$.

$$\Delta_n = \prod_{p \leq \Delta_n} p^{v_p(\Delta_n)}$$

or $p > n$, $v_p(\Delta_n) = 0$ donc $\Delta_n = \prod_{p \leq n} p^{v_p(\Delta_n)}$

: $p^{v_p(\Delta_n)} \leq n$

d'où $\boxed{\Delta_n \leq \prod_{p \leq n} n = n^{\pi(n)}}$

$\ln n^{\pi(n)} = \pi(n) \ln n \geq \ln \Delta_n \geq \ln 2^n$

(car $n \geq 7$)

Finalt $\boxed{\pi(n) \geq \ln 2 \cdot \frac{n}{\ln n}}$

$\pi(2) = 1 < 2$ (aïe)

$\pi(3) = 2 > 1{,}89$.

$\pi(4) = 2 \geq 2$

$\pi(5) = 3 > 2{,}153$

$\pi(6) = 3 > 2{,}32$

Finalt $\forall n \geq 2$

$$\pi(n) \geq \ln 2 \cdot \frac{n}{\ln n}$$